Deriving Link Travel Time Distributions via Stochastic Speed Processes

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Abstract

We derive an analytical expression for the cumulative distribution function of travel time for a vehicle traversing a freeway link of arbitrary length. The vehicle's speed is assumed to be modulated by a random environment which can be modeled as a stochastic process. We first present a partial differential equation (PDE) describing the travel time distribution and obtain a solution in terms of Laplace transforms. Next, we present a numerical inversion algorithm to invert the transforms. The technique is demonstrated on two example problems. Numerical results indicate great promise for this approach to the link travel time problem.

In this work, we introduce a new approach for analytically specifying the probability distribution of individual vehicle travel times for stochastic, time-varying transportation links such as unidirectional freeway segments. Knowledge of the link travel time distribution is extremely important in many areas of traffic analysis, namely those of real-time dynamic control of traffic systems and the least expected cost (time) routing of vehicles (e.g., emergency or hazardous material vehicles). In the optimal routing of vehicles through a transportation network, there exist several techniques to find the least time or least expected time paths when the link travel times are assumed to be deterministic or stochastic with time invariant distributions. In reality, link travel times are heavily dependent upon prevailing physical, traffic, and environmental conditions that cause the travel time to exhibit stochastic and time-variant behavior. Unfortunately, deterministic or estimated mean travel times may not be reliable input data for control or routing schemes due to potentially large variance in the travel time distribution for a given link. For these reasons, it is instructive to develop sound estimates for the time-dependent probability distribution of link travel times as input data to existing optimal control and routing algorithms (such as the least expected time algorithm of Miller-Hooks and Mahmassani, 2000).

One of the earliest studies dealing explicitly with the travel time distribution is that of Berry and Belmont (1951) who consider travel time distributions as they relate to the distribution of spot speeds, or the measured speed of a vehicle as it crosses a particular point on the highway. Such speed distributions were found to be normally distributed, except in the case where flow exceeds capacity. In such case, vehicle speeds tend to be skewed more toward higher velocities. Travel times, taken as the reciprocal of speed, are shown to also be roughly normal, although slightly skewed. However, this conclusion is valid only under the assumptions that individual vehicles maintain a constant speed and that speeds are symmetric about their mean. It is noted here that the first assumption is certainly not valid, especially in the case of moderate traffic. In addition, travel times will not be normally distributed when traffic volume exceeds link capacity.

Although some researchers have proposed a stochastic approach to the analysis of link travel times, none have considered an analytical specification of the travel time distribution. For instance, Jain and Smith (1997) model a freeway segment as a state-dependent M/G/C/C queue and calculate the steady-state performance measures such as mean time and number in system. The technique, however, does not provide the means for specifying the time-varying link travel time distribution analytically. Another trend in link travel time evaluation is in fitting different types of models to observed data at individual sites. For instance D'Angelo, *et al.* (1999) estimate link travel times using nonlinear time series analysis while Blue, *et al.* (1994) and Park, *et al.* (1998) apply artificial neural networks to attempt to predict travel times based on historical data. Although these approaches may be robust at the particular locations at which data is observed, they do not provide a general framework by which travel time may be evaluated apart from observed data. Furthermore, such techniques cannot be used to evaluate alternative designs for new facilities, or to analyze proposed changes to existing facilities.

Due to the availability of existing algorithms for finding the least expected time path for a transportation network, link travel time estimation is an extremely important topic. The problem is easily solved by standard techniques if link travel times are assumed to be deterministic or are made deterministic by using the expected value of random travel time when the distribution is assumed to be time-invariant. Such assumptions are usually deemed reasonable since most researchers have been more concerned with finding the shortest path through the network than with exact specification of link travel time distributions (Frank, 1969; Mirchandani and Soroush, 1985). As noted in Miller-Hooks and Mahmassani (2000), the problem becomes much more difficult (and realistic) when link travel times are assumed to be stochastic and time varying. The authors present algorithms to find the least expected time paths when arc weights (travel times) are chosen from a time-dependent, discrete probability distribution. It is assumed that such a distribution for each link (and in each time interval during a peak period) is given *a priori*; however, it is not clear how one would go about specifying such a distribution. This paper studies the link travel time independently of the network optimization problem in order to gain insight into the behavior of this quantity apart from network effects. By initially ignoring network effects, we are able to provide a significant (and necessary) first step for extending our approach to the overall network problem in future analysis.

We consider specification of the link travel time distribution by considering the stochastic elements that impact link travel time. The proposed technique considers the speed of a vehicle traversing a link as a finite-state Markov process that is modulated by a random environment process. The environment process models the physical, traffic, and environmental factors that are known to influence vehicle speed as the vehicle traverses a link in the transportation network. We show that, when the environment process is a continuous-time Markov chain (CTMC), an exact analytical expression is obtained for the Laplace-Stieltjes transform of the link travel time cumulative distribution function. We require as input, the infinitesimal generator matrix for the governing CTMC and a finite-dimensional vector of velocities. The generator matrix of the CTMC determines the rate at which speed transitions occur, while the environment-dependent velocities may be chosen as deterministic functions of the state of the environment. Our new approach is similar to the concept of stochastic fluid models which have enjoyed great success in the stochastic modeling of data communications networks (Anick, *et al.*, 1982; Elwalid and Mitra, 1993; Kesidis, *et al.*, 1993; Kulkarni and Gautam, 1997).

The main contributions of this work are a new approach for modeling the time-dependent speed of a vehicle traversing a network link and an explicit expression for the Laplace-Stieltjes transform of the link travel time distribution which can give approximate results with numerical inversion, or exact results when the transform can be algebraically inverted. Our approach implicitly captures the time dependence of vehicle speed by considering a random environment process that evolves stochastically over time for a given link of the transportation network. When the environment process is known to be a continuous-time Markov chain (CTMC), an explicit matrix equation for the link travel time distribution is obtained. The model can also be extended to more general environment processes such as semi-Markov or Markov regenerative processes, depending upon the application. The distribution of link travel times can be extremely valuable input in the analysis of transportation, communication, and manufacturing systems.

The remainder of the paper is organized as follows. The next section presents our analytical model for the travel time distribution. In Section 2, we use the analytical model to derive expressions for the travel time distribution. Section 3 discusses numerical inversion of the transform obtained in Section 2. Section 4 presents some numerical results on two example problems, while Section 5 explores strategies for obtaining the required inputs. Finally, our concluding remarks are given in Section 6.

1 Stochastic Model for Link Travel Time

Our main objective in this work is to derive an analytical expression for the cumulative distribution function (CDF) of travel time for an individual vehicle traversing a stochastic, time-varying transportation link, such as a unidirectional freeway segment. The difficulty in characterizing the travel time distribution for a particular vehicle stems from the fact that inherently stochastic processes govern the speed with which the vehicle may travel at a given point in time and space. Several factors influence the speed of the vehicle as it attempts to traverse the freeway. Some of those may be physical factors (e.g., roadway geometry, grades, visibility), traffic factors (e.g., density, presence of heavy vehicles, merging traffic), or environmental factors (e.g., weather conditions, speed limits, etc.).

In the early work of Berry and Belmont (1951), the authors stressed the need for research on the effect of physical, traffic, and environmental factors on vehicle speeds. Furthermore, Lighthill and Whitham (1955) noted that the classical continuous-fluid model could possibly represent the limiting behavior of underlying stochastic processes for infinitely long roadways. This raises an important question. Is it possible to determine the probability distribution of the travel time if the underlying stochastic processes governing the changes in speed are known? One approach might be to consider only the effect of arrivals and departures to the system. More specifically, a departure from the system downstream of the vehicle may allow the vehicle to increase its speed. An arrival upstream may catch up to the vehicle, thus increasing the density within its neighborhood and forcing it to assume a reduced speed. Downstream arrivals may also cause the vehicle to assume a decreased speed as the shock wave caused by the increased density propagates upstream. However, such an analysis would not account for physical or environmental factors involved in the travel time distribution. Furthermore, such an approach may be mathematically intractable unless restrictive assumptions are made. In the absence of a clear understanding of these processes, we propose a new approach wherein we assume that the speed of a vehicle is governed by a random environment and analyze the stochastic process of the speed to obtain the travel time distribution.

The approximate trajectory of a vehicle can be graphically depicted on a time-space diagram (cf. Coifman, 2002). Figure 1 gives an example of such a diagram in which $T(x)$ denotes the random time required to traverse a link of length x and $D(t)$ denotes the cumulative distance travelled after t time units.

PLEASE INSERT FIGURE 1 HERE

The true trajectory varies continuously over time and space as the vehicle assumes various speeds throughout its sojourn. The superimposed, piecewise-linear function approximates the true vehicle trajectory by assuming that the vehicle maintains a constant speed for a finite duration before making a speed transition. The slope of each chord corresponds to the speed of the vehicle. In this work, we assume that the vehicle's speed is governed by a random environment that may be modeled as a continuous-time stochastic process. The random environment process sojourns through a finite-dimensional state space, expending a random amount of time (the holding time) in any one of the states before making a transition to a different state. Corresponding to each state of the random environment is a particular speed (or range of speeds) that the vehicle may assume in the course of its sojourn. Such transitions in vehicle speed, due to external environmental factors, are particularly prevalent in congested periods wherein vehicles are unable to maintain their desired speeds (e.g., free flow speed). Initially, we assume that the random environment spends an exponentially distributed amount of time in a given state, thereby making the random environment process a continuous-time Markov chain (CTMC). It will have to be verified through empirical data if a CTMC is an accurate representation of the environment process. However, our methodology is a unique approach to the problem that allows us to analyze various aspects of travel time behavior. For this reason, we begin with this simplifying (exponential) assumption that lays the foundation for analyses involving more general environment processes. It should be noted that, in case empirical studies contradict the exponential assumption, it is still possible to model the environment process as a CTMC since any random variable may be represented as the sum of independent exponential random variables.

In accordance with the random environment process, the speed of a vehicle may now be considered as a continuous-time stochastic process whose continuous sample space is bounded below by zero and above by some appropriate value, such as the speed limit of the link or the free flow speed. Our approach here is to create a mapping between the conditions of the random environment and the speed of the vehicle with the overall objective of computing, analytically, the travel time of the vehicle. Hence, we discretize the continuous sample space of vehicle speed by creating a partition of the space. Suppose the lower bound on vehicle speed is V_1 and the speed limit (or free flow speed) is given by a value Λ. Define the continuous space of potential vehicle speeds as the closed interval $[V_1, \Lambda]$ miles per hour (mph). For an individual vehicle, we partition this set into a finite union of K disjoint sets as

$$[V_1, \Lambda] = \left(\bigcup_{i=1}^{K-1} [V_i, V_{i+1}) \right) \cup \{\Lambda\}, \tag{1}$$

where $V_K = \Lambda$. In our approach, the lower limit of the speed sample space, V_1, may be greater than or equal to zero. That is, in the course of its sojourn, the vehicle may assume speed zero which corresponds to a state of extreme traffic density (i.e., jam density). Figure 2 gives a graphical depiction of the discretization of the sample space of vehicle speed as it corresponds to the state of the random environment process.

PLEASE INSERT FIGURE 2 HERE

We now describe the relationship between the underlying random environment and vehicle speed.

Consider one link of a unidirectional freeway having physical length $x > 0$. The vehicle's speed at time $t > 0$ is governed by a random environment process, $\{Z(t) : t \geq 0\}$. The environment process has a finite state space, $S = \{1, 2, ..., K\}$, which implies that the vehicle may assume speeds in the finite set $\mathcal{V} = \{V_1, V_2, \ldots, V_K\}$ at any point in time. More specifically, when the environment is in state $i \in S$ (i.e., $Z(t) = i$), the vehicle assumes the speed V_i, the lower limit of the interval $[V_i, V_{i+1})$, at time t. We assume the lower limit in order to yield a conservative estimate of the overall travel time, though we are not restricted to this value. An arbitrary function $\zeta : S \to \mathcal{V}$ describes how the speed changes with regard to the environment. Hence, the time dependence of vehicle speed is captured implicitly through the environment process, $\{Z(t) : t \geq 0\}$, and the function ζ. It is obvious that the time of day at which a vehicle begins its sojourn will influence the stochastic time required to traverse the link. It is important to note that, with our technique, the current traffic conditions on the link are captured in the initial state of the environment at time zero, namely, $Z(0)$. In particular, we assume that, just before beginning the sojourn, the condition of the environment is known via the initial distribution of the environment process $\mathbf{z_0} = [P\{Z(0) = i\}]_{i \in S}$. Such information may be obtained through single or double-loop inductance detectors, surveillance cameras, probe vehicles, or other commonly used techniques in the transportation literature (cf. Hellinga and Gudapati, 2000).

Now let $T(x)$ be the continuous, random time required for a vehicle to traverse a freeway link with physical length x. The main objective of this paper is to analyze the distribution of $T(x)$, namely,

$$G(x;t) \equiv P\{T(x) \leq t\}. \tag{2}$$

We accomplish this objective by first deriving a partial differential equation describing the probability distribution of $D(t)$, the random distance travelled along the link up to time t, whose solution is obtained via transform methods. Next, we exploit a simple relationship between the distribution of $T(x)$ and $D(t)$. This is the subject of Section 2.

2 Analysis of the Model

In order to obtain the analytical travel time distribution, it is instructive to relate the distribution of $T(x)$ to the distribution of $D(t)$, which is explicitly a function of time. Lemma 1 describes the simple relationship that allows us to find the distribution of $T(x)$ by solving for the distribution of $D(t)$.

Lemma 1 *The events $\{D(t) \leq x\}$ and $\{T(x) \geq t\}$ are equivalent for all $x, t \geq 0$.*

Proof.

$$\begin{aligned} \{D(t) \leq x\} &\Leftrightarrow \text{ vehicle travels at most a distance } x \text{ up to time } t \\ &\Leftrightarrow \text{ it takes at least } t \text{ time units to travel a distance } x \\ &\Leftrightarrow \{T(x) \geq t\} \end{aligned}$$

■

Using Lemma 1, it is easy to see that

$$G(x;t) \equiv P\{T(x) \leq t\} = 1 - P\{D(t) \leq x\}, \quad x, t \geq 0. \tag{3}$$

Now define the following joint probability distribution,

$$H_i(x,t) = P\{D(t) \leq x, Z(t) = i\}, \quad i \in S \tag{4}$$

where $H_i(x,t)$ is the joint probability that, at time t, the vehicle has travelled a distance no greater than x and the environment process is in state $i \in S$. The objective is to find the joint distribution $H_i(x,t)$ and apply it in Equation (3) to find the travel time distribution. More specifically,

$$
\begin{aligned}
G(x;t) \equiv P\{T(x) \leq t\} &= 1 - P\{D(t) \leq x\} \\
&= 1 - \sum_{i \in S} H_i(x,t). \qquad (5)
\end{aligned}
$$

In this research, it is assumed that the random environment process is a continuous-time Markov chain (CTMC). As previously noted, we employ this simplifying assumption in order to obtain closed-form, analytical results for the travel time distribution that will provide insight into the behavior of the link travel time that would not otherwise be available. Furthermore, if it is found that the CTMC assumption is not valid, then appropriate modifications can be made to make the following analysis applicable. The CTMC assumption leads to the following theorem, the main result of this paper.

Theorem 1 *If the random environment process $\{Z(t) : t \geq 0\}$ governing vehicle speed is a continuous-time Markov chain with infinitesimal generator, $Q = [q_{ij}]$, then $H_i(x,t)$ satisfies the partial differential equation*

$$
\frac{\partial H_i(x,t)}{\partial t} + \frac{\partial H_i(x,t)}{\partial x} V_i = \sum_{j \in S} q_{ji} H_i(x,t), \qquad i \in S \qquad (6)
$$

with initial condition

$$
H_i(x,0) = B_i(x) = P\{Z(0) = i\}.
$$

Proof. Let $h > 0$.

$$
\begin{aligned}
H_i(x,t+h) &= P\{D(t+h) \leq x, Z(t+h) = i\} \\
&= \sum_j P\{Z(t+h) = i | D(t+h) \leq x, Z(t) = j\} \\
&\quad \times P\{D(t+h) \leq x | Z(t) = j\} P\{Z(t) = j\} \\
&= (1 + q_{ii}h) H_i(x - V_i h, t) + \sum_{j \neq i} q_{ji} h H_j(x - V_j h, t) + o(h)
\end{aligned}
$$

Simplifying and dividing by the time increment, h, yields,

$$
\frac{H_i(x,t+h) - H_i(x,t)}{h} = \frac{H_i(x - V_i h, t) - H_i(x,t)}{h} + \sum_{j \in S} q_{ji} H_j(x - V_j h, t) + \frac{o(h)}{h}.
$$

Letting $h \downarrow 0$ gives

$$\frac{\partial H_i(x,t)}{\partial t} + \frac{\partial H_i(x,t)}{\partial x} V_i = \sum_{j \in S} q_{ji} H_i(x,t). \tag{7}$$

■

Now, writing Equation (7) in matrix form we have

$$\frac{\partial H(x,t)}{\partial t} + \frac{\partial H(x,t)}{\partial x} V = H(x,t)Q \tag{8}$$

where $H(x,t) = [H_i(x,t)]_{i \in S}$ is the $1 \times K$ row vector of joint distribution values and $V \equiv diag(V_1, V_2, \ldots, V_K)$. Solving Equation (8) is not a trivial task; thus, we employ transform techniques to obtain an approximate solution. To that end, denote by $H_i^*(x, s_2)$, the Laplace transform (LT) of $H_i(x,t)$ with respect to t. Let $\tilde{H}_i^*(s_1, s_2)$, $i \in S$ denote the Laplace-Stieltjes transform (LST) of $H_i^*(x, s_2)$ with respect to x, $i \in S$ and define $\tilde{H}^*(s_1, s_2) = [\tilde{H}_i^*(s_1, s_2)]_{i \in S}$ as the $1 \times K$ row vector of transforms. Theorem 2 provides the means by which to compute the transform equations.

Theorem 2 *The solution to the differential equation (6) in the transform space is given by*

$$\tilde{H}^*(s_1, s_2) = \tilde{B}(s_1)(s_1 V + s_2 I - Q)^{-1} \tag{9}$$

where $H^*(x, s_2) = [H_i^*(x, s_2)]_{i \in S}$ is the row vector of Laplace transforms of $H_i(x,t)$ with respect to t, $\tilde{B}(s_1)$ is a $1 \times K$ row vector and s_1 and s_2 are complex transform variables with $Re(s_1) > 0$ and $Re(s_2) > 0$.

Proof. Taking the LT of both sides of Equation (8) with respect to t, yields

$$-H(x,0) + s_2 H^*(x, s_2) + \frac{\partial H^*(x, s_2)}{\partial x} V = H^*(x, s_2)Q.$$

Now, taking the Laplace-Stieltjes transform of the above equation with respect to x gives

$$-\tilde{B}(s_1) + s_2 \tilde{H}^*(s_1, s_2) + s_1 \tilde{H}^*(s_1, s_2)V = \tilde{H}^*(s_1, s_2)Q, \tag{10}$$

Rearranging the terms of Equation (10) yields the desired result. ■

Theorem 2 gives an analytical expression for the transform that may be inverted twice to obtain the distribution of $T(x)$, the random time to travel a distance x. In general, an exact expression for the inverse transform is available when Equation (9) is a vector of rational functions in both of the complex variables, s_1 and s_2. In such case, the inversion may be done via partial fractions. However, this will be a cumbersome task if $K = |S|$ is large. For these reasons, we employ numerical inversion techniques to find approximate solutions for Equation (8). In Section 3 , we present a numerical inversion technique that may be used to accomplish this task. First, we review some preliminaries of numerical inversion of Laplace transforms in one and two dimensions.

3 Numerical Inversion

Inversion of Equation (9) is not an easy task. In this section, the numerical inversion of one and two-dimensional Laplace transforms is reviewed. Transformation of the joint distribution into the frequency domain allows for an exact solution to the partial differential equation if the two-dimensional transform can be inverted. Recall that the Laplace transform of a real function of time, $f(t)$ denoted by $\mathcal{L}(f(t)) = f^*(s)$ is

$$\mathcal{L}(f(t)) = f^*(s) = \int_0^\infty e^{-st} f(t) dt, \tag{11}$$

where s is the complex transform variable and $Re(s) > 0$ (the real part of s is positive). The inverse exists if and only if the function f is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty. \tag{12}$$

Recovery of the time-domain function, f, may be accomplished by using the Laplace transform tables if the transform, f^*, can be expressed as a ratio of two rational polynomials in the complex variable $s = c + jd$, where c and d are the real and imaginary parts of s, respectively. Otherwise, recovery of the original function is accomplished through the inverse transform

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{st} f^*(s) ds, \tag{13}$$

which is usually solved by numerical methods for an approximate solution. Numerical inversion for a transform in one variable has been studied extensively in the literature. Most notable are the techniques of Weeks (1966), Dubner and Abate (1968), and Abate, *et al.* (1996). The techniques of Weeks (1966) and Abate, *et al.* (1996) utilize the Laguerre series representation of f due to the fact that the Laguerre generating function may be expressed directly in terms of f^*, the Laplace transform. Abate, *et al.* (1996) improve upon Weeks' approach by implementing a scaling technique and accelerating convergence by the $\epsilon-$algorithm (MacDonald, 1964; Wynn, 1966). The method of Dubner and Abate (1968) uses Fourier series representations which are noted for their accuracy on a variety of function types.

In the travel time distribution problem, the original function, $H_i(x,t)$, may be considered as a two-dimensional function, even though the travel time is for a fixed distance, x. Ditkin and Prudnikov (1962) give an excellent treatment of the operational calculus in two dimensions, and the major concepts are reviewed here. For a real-valued function of two continuous variables, x and t, the two-dimensional Laplace transform of f is given by

$$\mathcal{L}(f(x,t)) = f^*(s_1,s_2) = \int_0^\infty \int_0^\infty e^{-(s_1x+s_2t)} f(x,t)\,dx\,dt, \tag{14}$$

with inverse transform

$$f(x,t) = \frac{1}{4\pi^2} \int_{c_1-j\infty}^{c_1+j\infty} \int_{c_2-j\infty}^{c_2+j\infty} e^{(s_1x+s_2t)} f^*(s_1,s_2)\,ds_1\,ds_2. \tag{15}$$

The inverse transform equation requires approximation of the integrand in order to perform the double numerical integration with respect to the two complex variables, $s_1 = c_1 + jd_1$ and $s_2 = c_2 + jd_2$. A number of algorithms have been proposed for the numerical inversion of two-dimensional Laplace transforms. Of particular interest are the techniques of Choudhury, *et al.* (1994) and Moorthy (1995). An earlier technique due to Singhal, *et al.* (1975) uses Padé approximations. More recently, Abate, *et al.* (1998) expand the Laguerre method to the two-dimensional case and demonstrate the efficacy of their procedure on applied probability problems.

We focus here on the techniques of Choudhury, *et al.* (1994) and Moorthy (1995). Both papers utilize the Fourier series representation of the original function f. Moorthy (1995) in particular extends the work of Dubner and Abate (1968) to the case of two dimensions. Choudhury, *et al.* (1994), present an approach very similar to that of Moorthy while incorporating additional techniques to control for discretization, truncation, and roundoff error. In this work, we utilize ideas from both papers to invert the transforms of Equation (9).

Following the notation of Moorthy (1995), an approximation for the inverse function, $f(x,t)$, is given by

$$f(x,t) \approx \bar{f}(x,t) = (1/2)\exp(c_1 x + c_2 t)(T^{-2})\left\{f^*(c_1,c_2)/2 + \sum_{i=1}^{3} k_i\right\} \tag{16}$$

where

$$\begin{aligned}
k_1 &= \sum_{m=1}^{\infty} Re\{f^*(c_1, c_2 + i\pi m/T)\}\cos(m\pi t/T) - Im\{f^*(c_1, c_2 + i\pi m/T)\}\sin(m\pi t/T)\\
k_2 &= \sum_{n=1}^{\infty} Re\{f^*(c_1 + i\pi n/T, c_2)\}\cos(n\pi x/T) - Im\{f^*(c_1 + i\pi n/T, c_2)\}\sin(n\pi x/T)\\
k_3 &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Re\{f^*(c_1 + i\pi n/T, c_2 + i\pi m/T)\}\cos(\pi n x/T + \pi m t/T)\\
&\quad +Re\{f^*(c_1 + i\pi n/T, c_2 - i\pi m/T)\}\cos(\pi n x/T - \pi m t/T)\\
&\quad -Im\{f^*(c_1 + i\pi n/T, c_2 + i\pi m/T)\}\sin(\pi n x/T + \pi m t/T)\\
&\quad -Im\{f^*(c_1 + i\pi n/T, c_2 - i\pi m/T)\}\sin(\pi n x/T - \pi m t/T)\\
c_1 &= A_1/2xl_1\\
c_2 &= A_2/2tl_2
\end{aligned}$$

The technique assumes f is periodic with period $2T$. The parameters, A_i and $l_i, i = 1, 2$ are chosen as per Choudhury, *et al.* (1994). $Re(w)$ and $Im(w)$ denote the real and imaginary parts of the complex number w, respectively. It is important to note that concepts from both papers are used here. We use the Fourier series representation of Moorthy(1995) for the inverse function, $H_i(x,t)$. Careful examination of both techniques indicates the equivalence of Moorthy's parameters c_i, $i = 1, 2$ and a_i, $i = 1, 2$ of Choudhury, *et al.* (1994). Thus we select the parameters, A_i and $l_i, i = 1, 2$ of Choudhury, *et al.* (1994) to control the discretization and roundoff error, and use the ϵ-algorithm to accelerate convergence.

As previously noted, the technique of Moorthy (1995) assumes that the function, $f(x, t)$ is periodic with period $2T$. This is generally not a problem as long as T is appropriately chosen. Let $T_{max} \equiv \max\{x, t\}$ and select T such that

$$0.5T_{max} \leq T \leq 0.8T_{max}.$$

One pragmatic choice for the parameter T is the mid-point of the interval, $0.65T_{max}$. In Section 3.1, a formal description of the algorithm is provided.

3.1 Description of Inversion Algorithm

Step 0: *Parameter selection.* Since the function will be a cumulative probability, select the parameters $A_1 = A_2 = 28.324$ and $l_1 = l_2 = 3$ in accordance with the guidelines of Choudhury, *et al.* (1994). The parameters c_1 and c_2 are calculated as per Equation (16) and let $T = 0.65T_{max}$.

Step 1: *Compute infinite sums of Equation (16) using the ϵ-algorithm.* The ϵ-algorithm requires the selection of two parameters, m and n. The approximation for an infinite series is constructed by solving the recursive equation

$$\epsilon_{k+1}^{n} = \epsilon_{k-1}^{n+1} + (\epsilon_{k}^{n+1} - \epsilon_{k}^{n})^{-1}$$

where $\epsilon_{-1}^{n} = 0$ and ϵ_{0}^{n} is the n^{th} partial sum of the infinite series. The final approximation of the infinite series is ϵ_{2m}^{n}. The values $m = 6$ and $n = 24$ will be adequate.

Step 2: *Compute $\bar{f}(x, t)$ by Equation (16).* It is important to note that the inversion algorithm is expected to perform extremely well whenever the inverse transform, $f(x, t)$ decays exponentially and is sufficiently smooth.

4 Numerical Results

In this section, we present two numerical examples in which we obtain the travel time distribution for a vehicle whose speed is modulated by a continuous-time Markov chain. The first example is a two-state CTMC while the second example involves a CTMC with 5 states.

Example 1

In this example, we partition the speed sample space into two intervals so that the vehicle assumes two velocities in accordance with a two-state environment process having state space $S = \{1, 2\}$. When the environment process is in state i, the vehicle has speed V_i, $i = 1, 2$. The velocities are chosen such that speed $V_1 > V_2$. The amount of time spent in state 1 is exponentially distributed with rate β, while the time spent in state 2 is exponentially distributed with rate α. We arbitrarily assume that, with probability 1, the system starts in state 1 at time 0. That is,

$$P\{Z(0) = 1\} = 1 \Rightarrow \tilde{B}(s_1) = [\ 1 \quad 0\ ].$$

The relevant matrices for this system are

$$Q = \begin{bmatrix} -\beta & \beta \\ \alpha & -\alpha \end{bmatrix}$$

and

$$V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}.$$

Now, the partial differential equation describing the probability distribution of $D(t)$ is given by

$$\frac{\partial H_i(x,t)}{\partial t} + \frac{\partial H_i(x,t)}{\partial x} V_i = \sum_{j=1}^{2} q_{ji} H_i(x,t). \qquad i = 1, 2$$

or in matrix form,

$$\frac{\partial}{\partial t}\begin{bmatrix} H_1(x,t) \\ H_2(x,t) \end{bmatrix}^T + \frac{\partial}{\partial x}\begin{bmatrix} H_1(x,t) \\ H_2(x,t) \end{bmatrix}^T \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} = \begin{bmatrix} H_1(x,t) \\ H_2(x,t) \end{bmatrix}^T \begin{bmatrix} -\beta & \beta \\ \alpha & -\alpha \end{bmatrix},$$

which can be written as

$$\frac{\partial H(x,t)}{\partial t} + \frac{\partial H(x,t)}{\partial x} V = H(x,t)Q,$$

where $H(x,t) = [H_i(x,t)]_{i=1,2}$. Now applying Theorem 2, the matrix $\tilde{H}^*(s_1, s_2)$ is obtained by

$$\begin{aligned} \tilde{H}^*(s_1, s_2) &= \tilde{B}(s_1)(s_1 V + s_2 I - Q)^{-1} \\ &= [\ 1 \quad 0\ ](\delta\gamma - \alpha\beta)^{-1} \begin{bmatrix} \delta & \beta \\ \alpha & \gamma \end{bmatrix}, \end{aligned}$$

where $\delta = s_1 V_2 + s_2 + \alpha$ and $\gamma = s_1 V_1 + s_2 + \beta$ are scalars. In order to implement the inversion algorithm of Section 3.1, we convert the LST above into a LT by pre-multiplying the vector $\tilde{B}(s_1)$ by s_1^{-1} so that

$$\hat{H}(s_1, s_2) = [\ s_1^{-1} \quad 0\ ](\delta\gamma - \alpha\beta)^{-1} \begin{bmatrix} \delta & \beta \\ \alpha & \gamma \end{bmatrix}.$$

The notation $\hat{H}(s_1, s_2) = [\hat{H}_i(s_1, s_2)]_{i=1,2}$ is used for the two-dimensional Laplace transform in x and t. The parameters chosen for this problem are $\alpha = \beta = 500\ hr^{-1}$, $V_1 = 65$ mph, and $V_2 = 15$ mph. The inversion algorithm of Section 3.1 was implemented using the MATLAB mathematical computing package. In order to test the adequacy of the inversion algorithm, an empirical CDF based on 100,000 observations of travel time was generated via Monte-Carlo simulation methods. For each observation, the program simulates the speed process (over time) until the cumulative distance travelled by a vehicle is x miles. In this example, we fixed x at 1.0 mile. The results of Table 1 confirm that the inversion algorithm gives very acceptable results for this problem.

PLEASE INSERT TABLE 1 HERE

Example 2

In this example, the random environment process is a general, five-state CTMC with state space $S = \{1, 2, 3, 4, 5\}$. When the environment is in state i, the speed of the vehicle is $V_i = 75/i$ for $i \in S$. The off-diagonal entries of the generator matrix, $Q = [q_{ij}]_{i,j \in S}$, are obtained by sampling from a uniform distribution on the interval (200,400) hr^{-1}. If the CTMC is currently in state $i \in S$, the process has a probability of $q_{ij}/(-qii)$ of transitioning to state $j \in S \setminus \{i\}$. It is arbitrarily assumed that, with probability 1, the system starts in state 1 at time 0 so that

$$P\{Z(0) = 1\} = 1 \Rightarrow \tilde{B}(s_1) = [\ 1 \quad 0 \quad 0 \quad 0 \quad 0\ ].$$

The off-diagonal entries of Q were computed by generating a uniform variate on the interval (0,1) and translating each entry so that it lies in the interval (200,400), which is chosen arbitrarily. The infinitesimal generator for this problem is generated randomly as

$$Q = \begin{bmatrix} -919.75 & 206.91 & 264.85 & 238.67 & 209.32 \\ 223.01 & -971.71 & 301.98 & 232.73 & 213.98 \\ 343.04 & 277.78 & -1283.57 & 392.72 & 270.03 \\ 353.91 & 232.27 & 213.69 & -1059.47 & 259.59 \\ 370.92 & 200.89 & 216.80 & 225.60 & -1014.21 \end{bmatrix}$$

One may now apply Theorem 2 in a manner similar to that of Example 1 to solve for the distribution. As before, an empirical CDF based on 100,000 observations of travel time was generated via Monte-Carlo simulation methods to validate the inversion algorithm. The length of the link was set to $x = 1.0$ mile and $G(x;t) = P\{T(x) \leq t\}$ was computed for a number of t-values. The results of Table 2 confirm that the inversion algorithm gives very accurate results, even when the CTMC is chosen in an arbitrary manner.

PLEASE INSERT TABLE 2 HERE

In Section 5, we discuss strategies for obtaining parameters of the infinitesimal generator matrix, Q.

5 Obtaining the Generator Matrix

In order to implement the methods of this paper within a real-world context, two components are needed, namely the infinitesimal generator matrix Q and the initial probability distribution of the environment process $\{Z(t) : t \geq 0\}$. Specification of the matrix Q entails approximation of the rate at which environment (and thus speed) transitions occur. Let $q_i > 0$ denote the total rate at which the environment makes transitions out of state i. Furthermore, let q_{ij}, the $(i,j)^{th}$ element of Q, denote the rate at which the vehicle transitions from state $i \in S$ to $j \in S$ such that $q_i = -q_{ii} = \sum_{j \neq i} q_{ij}$. We now describe a simple method for obtaining statistical estimates of these parameters, namely $\hat{q}_{ij}$ $\forall i, j \in S$ for an individual link.

By tagging individual vehicles, or through the use of on-board instrumentation, it is possible to track individual vehicles during the course of their sojourn over a particular link. By recording the observed vehicle range and duration of time spent in that range, one may construct a table of values that can be used to approximate all the elements of the Q matrix. In particular, one may draw four columns for a vehicle to record the pertinent data. Table 3 displays sample data that might be observed in a real scenario.

PLEASE INSERT TABLE 3 HERE

Using such a table, the data can be sorted according to the second column and a nonparametric density estimate (such as a simple histogram) can be drawn for the amount of time spent in each speed range. If the histograms exhibit exponential behavior, then the approach of this paper may be directly applied. More specifically, if the mean and standard deviation of the duration in each speed range are roughly equal, then the total rate of leaving state i, q_i can be estimated by the reciprocal of the mean time in state i denoted by $\hat{q}_i$. Define by $\hat{p}_{ij}$, the fraction of transitions out of state i into state j. The off-diagonal elements of Q can then estimated by

$$\hat{q}_{ij} = \hat{p}_{ij}\hat{q}_i. \tag{17}$$

The initial distribution of the environment process, $P\{Z(0) = i\}$, $i = 1, 2, \ldots, K$ can also be easily obtained through the observed data given above, or by estimating the initial speed of the vehicle through standard techniques, such as the use of single loop detectors at the entry point of the link. Thereby, if the initial speed range is observed to be state i, then we set,

$$P\{Z(0)=j\}=\begin{cases}1 & j=i\\ 0 & j\neq i\end{cases} \tag{18}$$

When computing the total link travel time distribution, the methods above hold several advantages over other potential approaches, such as simply tagging vehicles and recording travel times or discrete-event simulation. First, our approach requires less time to construct the distribution. This may be significant if the distribution changes rapidly with time. Second, once the Q matrix has been approximated, significant computational savings can be realized over simulation experiments. For instance, the inversion algorithm computed the travel time distribution six times faster than a Monte Carlo simulation experiment for a problem involving 10 distinct environment states. Lastly, the analytical specification of the travel time distribution gives us insight into the nature of the travel time random variable that will enable us to study the moments of travel time, asymptotic behavior and other properties.

The approach here may be adapted to an entire path through a transportation network. This may be accomplished by assigning to each link along a path, a distinct generator matrix describing the behavior of environment transitions. Hence, the final state of the first link becomes the initial state of the subsequent link along the path. Analytical methods for describing this hand-off interaction between two adjacent links are currently being explored.

6 Conclusions and future work

In this work, we have introduced a new approach for analytically specifying the probability distribution of individual vehicle travel times for stochastic, time-varying transportation links such as unidirectional freeway segments. Such information can be extremely valuable input data to existing algorithms for the real-time, dynamic control of traffic systems and the least cost (time) routing of vehicles. The technique considers the speed of a vehicle as a finite-state Markov process that is modulated by a random environment. The environment process models the physical, traffic, and environmental factors that are known to influence vehicle speed as the vehicle traverses a link in the transportation network. We have shown that, when the environment process is a continuous-time Markov chain (CTMC), an exact analytical expression is obtained for the Laplace-Stieltjes transform of the link travel time cumulative distribution function. The technique requires as input, only the infinitesimal generator matrix for the governing CTMC, a finite-dimensional vector of velocities, and specification of the initial state of the environment. The generator matrix of the CTMC determines the rate at which speed transitions occur, while the environment-dependent velocities may be chosen as deterministic functions of the state space. Strategies for selecting the appropriate generator matrix were given in Section 5 when the CTMC assumption can be employed. However, even if empirical data does not support this assumption, appropriate modifications can be made to approximate the state holding times by sums of exponential random variables.

The stochastic process for vehicle speed leads to a partial differential equation that can be solved via transform methods and algebraic (numerical) inversion techniques for exact (approximate) results. An algorithm was presented for the numerical inversion of the matrix equation and was shown to perform well on two example problems. By using the analytical results of this paper, it may now be possible to obtain all the moments of the random travel time for the purpose of constructing surrogate distribution approximations that can be computed with far less computational effort than the methods presented or through Monte Carlo simulation methods.

Although the technique presented herein provides an analytical result for the link travel time distribution, some ingenuity will be required in determining the appropriate selection of transition rates for the CTMC (as described in Section 5) and the speed function, $\zeta(\cdot)$. The properties of the speed function should be in accordance with well-established traffic flow relationships. For instance, if the states of the environment process correspond to the number of vehicles present on a link upon arrival, then the speed is probably best modeled as an exponentially decaying function of the number of vehicles present (Jain and Smith, 1997). It will be instructive in the future to derive explicit expressions for the moments of the random travel time and asymptotic approximations thereof. Furthermore, we would next like to consider the travel time distribution over adjacent links governed by distinct random environments. In addition to transportation links, the work presented herein is equally applicable in the modeling and analysis of stochastic, time-varying data communication or manufacturing system links.

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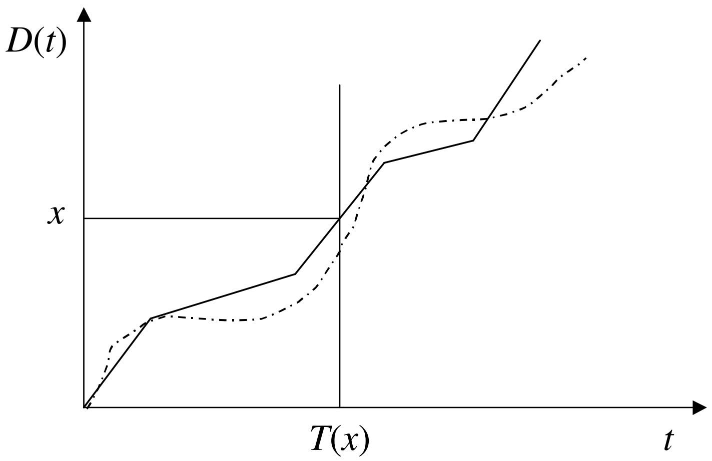

Figure 1: Estimated trajectory of vehicle using piece-wise, linear functions.

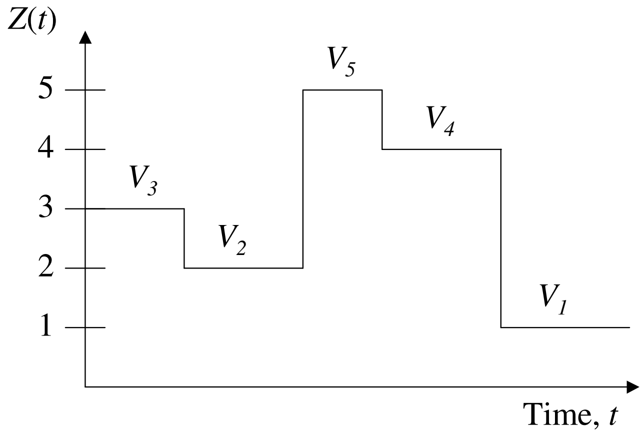

Figure 2: A sample path of the environment process, $\{Z(t) : t \geq 0\}$.

Table 1: $G(x;t)$ for the 2-state CTMC, $\alpha = \beta = 500$, $V_1 = 65, V_2 = 15$.

x (mi)	t (min)	*Inversion*	*Simulation*	*% Error*
1.0	1.20	0.1259	0.1288	2.260
1.0	1.29	0.2373	0.2430	2.336
1.0	1.38	0.3720	0.3704	0.420
1.0	1.47	0.5128	0.5140	0.225
1.0	1.56	0.6437	0.6484	0.731
1.0	1.65	0.7539	0.7501	0.504
1.0	1.74	0.8396	0.8360	0.431
1.0	1.84	0.9010	0.9002	0.091
1.0	1.93	0.9420	0.9424	0.037
1.0	2.02	0.9677	0.9698	0.221
1.0	2.11	0.9830	0.9829	0.009
1.0	2.20	0.9915	0.9914	0.009
1.0	2.29	0.9958	0.9957	0.014
1.0	2.38	0.9982	0.9980	0.019
1.0	2.47	0.9991	0.9994	0.028
1.0	2.56	0.9995	0.9996	0.010
1.0	2.66	0.9999	0.9996	0.029
1.0	2.75	1.0000	1.0000	0.003

Table 2: $G(x;t)$ for a general, five-state CTMC.

x (mi)	t (min)	*Inversion*	*Simulation*	*% Error*
1.0	1.25	0.0786	0.0777	1.134
1.0	1.47	0.3335	0.3352	0.497
1.0	1.70	0.6859	0.6865	0.086
1.0	1.92	0.9141	0.9136	0.059
1.0	2.14	0.9873	0.9872	0.012
1.0	2.37	0.9991	0.9991	0.005
1.0	2.59	1.0000	0.9999	0.002
1.0	2.81	1.0000	1.0000	0.000

Table 3: Notional data for generator estimation procedure.

Vehicle no.	Speed Range (mph)	Duration (sec)	Next Range (mph)
1	10-20	28.889	20-30
1	20-30	6.802	30-40
1	30-40	12.832	40-50
1	50-60	14.756	60-70
2	20-30	6.802	30-40
2	30-40	12.832	40-50
2	50-60	14.756	60-70
2	40-50	57.409	50-60
3	10-20	21.235	20-30
3	20-30	23.520	30-40
3	30-40	44.225	40-50
3	50-60	10.203	50-60
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